

Measure and mass gap for generalized connections on hypercubic lattices

R. Vilela Mendes *

Abstract

Using projective limits as subsets of Cartesian products of homomorphisms from a lattice to the structure group, a consistent interaction measure and an infinite-dimensional calculus has been constructed for a theory of non-abelian generalized connections on a hypercubic lattice. Here, after reviewing and clarifying past work, new results are obtained for the mass gap when the structure group is compact.

1 Introduction

In [1] a space for generalized connections was defined using projective limits as subsets of Cartesian products of homomorphisms from a lattice to a structure group. In this space, non-interacting and interacting measures were defined as well as functions and operators. From projective limits of test functions and distributions on products of compact groups, a projective gauge triplet was obtained, which provides a framework for an infinite-dimensional calculus in gauge theories.

In [1] a central role is played by the construction of an interacting measure which, satisfying a consistency condition, can be extended to a projective limit of decreasing lattice spacing and increasingly larger lattices. Since [1] was published some questions have been raised concerning in particular the construction of the measure and the consistency condition. The purpose of

*CMAF, Universidade de Lisboa, Av. Gama Pinto, 2 - 1649-003 Lisboa (Portugal),
rvmendes@fc.ul.pt, rvilela.mendes@gmail.com

this paper is twofold. First to clarify and extend some details of the measure construction which, of course, were implicit in [1]. Second to further explore some of the physical consequences of the constructed measure, in particular the nature of the mass gap that it implies.

The basic setting, as used in [1], is the following:

In \mathbb{R}^4 a sequence of hypercubic lattices is constructed in such a way that any plaquette of edge size $\frac{a}{2^k}$ ($k = 0, 1, 2, \dots$) is a refinement of a plaquette of edge $\frac{a}{2^{k-1}}$ (meaning that all vertices of the $\frac{a}{2^{k-1}}$ plaquette are also vertices in the $\frac{a}{2^k}$ plaquettes). The refinement is made one-plaquette-at-a-time, in the sense that, when one plaquette of edge $\frac{a}{2^{k-1}}$ is converted into four plaquettes of edge $\frac{a}{2^k}$, eight new plaquettes of edge $\frac{a}{2^{k-1}}$, orthogonal to the refined plaquette, are also added to the lattice. The additional plaquettes connect the new vertices of the refined $\frac{a}{2^k}$ plaquette to the middle points of $\frac{a}{2^{k-1}}$ plaquettes, in such a way that when all $\frac{a}{2^{k-1}}$ plaquettes are refined to $\frac{a}{2^k}$ size, a full hypercubic $\frac{a}{2^k}$ lattice is obtained. See Fig.1 for a 3-dimensional projection of the process, where two of the additional eight (in \mathbb{R}^4) plaquettes are shown, attached to the points A, B, C and D . This one-plaquette-at-a-time construction is useful to check the consistency condition (see Section 2).

Finite volume hypercubes Γ in these lattices form a directed set $\{\Gamma, \succ\}$ under the inclusion relation \succ . $\Gamma \succ \Gamma'$ meaning that all edges and vertices in Γ' are contained in Γ , the inclusion relation satisfying

$$\begin{aligned} \Gamma &\succ \Gamma \\ \Gamma &\succ \Gamma' \text{ and } \Gamma' \succ \Gamma \implies \Gamma = \Gamma' \\ \Gamma &\succ \Gamma' \text{ and } \Gamma' \succ \Gamma'' \implies \Gamma \succ \Gamma'' \end{aligned} \tag{1}$$

After each complete refinement of a finite volume hypercube (from $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$ size), the sequence is expanded to include larger and larger volume hypercubes which are likewise refined, etc..

Let \mathbb{G} be a compact group and x_0 a point that does not belong to any lattice point of the directed family. Assuming an analytic parametrization of each edge, associate to each edge l a x_0 -based loop and for each generalized connection A consider the holonomy $h_l(A)$ associated to this loop. For definiteness each edge is considered to be oriented along the coordinates positive direction and the set of edges of the lattice Γ is denoted $E(\Gamma)$. The set \mathcal{A}_Γ of generalized connections for the lattice hypercube Γ is the set of homomorphisms $\mathcal{A}_\Gamma = \text{Hom}(E(\Gamma), G) \sim G^{\#E(\Gamma)}$, obtained by associating to each edge the holonomy $h_l(\cdot)$ on the associated x_0 -based loop. The set of

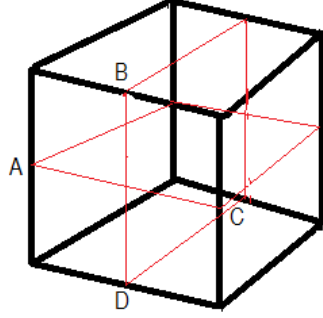


Figure 1: Partial 3-dimensional projection of the one-plaquette-at-a-time refinement process

gauge-independent generalized connections \mathcal{A}_Γ/Ad is obtained factoring by the adjoint representation at p_0 , $\mathcal{A}_\Gamma/Ad \sim G^{\#E(\Gamma)}/Ad$. However because, for gauge independent functions, integration in \mathcal{A}_Γ coincides with integration in \mathcal{A}_Γ/Ad , for simplicity, from now on one uses only \mathcal{A}_Γ . Finally one considers the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_\Gamma$ of the family

$$\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'} : \Gamma' \succ \Gamma\} \quad (2)$$

$\pi_{\Gamma\Gamma'}$ and π_Γ denoting the surjective projections $\mathcal{A}_{\Gamma'} \rightarrow \mathcal{A}_\Gamma$ and $\mathcal{A} \rightarrow \mathcal{A}_\Gamma$.

The projective limit of the family $\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'}\}$ is the subset \mathcal{A} of the Cartesian product $\prod_\Gamma \mathcal{A}_\Gamma$ defined by

$$\mathcal{A} = \left\{ a \in \prod_\Gamma \mathcal{A}_\Gamma : \Gamma' \succ \Gamma \implies \pi_{\Gamma\Gamma'} a_{\Gamma'} = a_\Gamma \right\} \quad (3)$$

the projective topology in \mathcal{A} being the coarsest topology for which each π_Γ mapping is continuous.

For a compact group \mathbb{G} , each \mathcal{A}_Γ is a compact Hausdorff space. Then \mathcal{A} is also a compact Hausdorff space. In each \mathcal{A}_Γ one has a natural (Haar)

normalized product measure $\nu_\Gamma = \mu_H^{\#E(\Gamma)}$, μ_H being the normalized Haar measure in \mathbb{G} . Then, according to a theorem of Prokhorov, as generalized by Kisynski [2] [3], if

$$\nu_{\Gamma'}(\pi_{\Gamma\Gamma'}^{-1}(B)) = \nu_\Gamma(B) \quad (4)$$

for every $\Gamma' \succ \Gamma$ and every Borel set B in \mathcal{A}_Γ , there is a unique measure ν in \mathcal{A} such that $\nu(\pi_\Gamma^{-1}(B)) = \nu_\Gamma(B)$ for every Γ .

2 The measure

As stated before, the essential step in the construction of the measure in the projective limit is the fulfilling of the consistency condition (4). One considers, on the finite-dimensional spaces $\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}$, measures that are absolutely continuous with respect to the Haar measure

$$d\mu_{\mathcal{A}_\Gamma} = p(\mathcal{A}_\Gamma) (d\mu_H)^{\#E(\Gamma)} \quad (5)$$

$p(\mathcal{A}_\Gamma)$ being a continuous function in \mathcal{A}_Γ with the simplifying assumptions:

- $p(\mathcal{A}_\Gamma)$ is a product of plaquette functions

$$p(\mathcal{A}_\Gamma) = p(U_{\square_1}) p(U_{\square_2}) \cdots p(U_{\square_n}) \quad (6)$$

with $U_\square(\mathcal{A}_\Gamma) = h_1 h_2 h_3^{-1} h_4^{-1}$, h_1 to h_4 being the holonomies of the x_0 -based loops associated to the edges of the plaquette.

- $p(\cdot)$ is a central function, $p(xy) = p(yx)$ or, equivalently $p(y^{-1}xy) = p(x)$ with $x, y \in \mathbb{G}$.

Let p', p'' and p be the density functions associated respectively to the square plaquette with edges of size $\frac{a}{2^k}$, to the rectangular plaquette with edges of size $\frac{a}{2^k}$ and $\frac{a}{2^{k-1}}$ and, finally, to the square plaquette with edges of size $\frac{a}{2^{k-1}}$. Then

Theorem 1 [1] *A measure on the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_\Gamma$ exists if a sequence of functions is found satisfying*

$$\begin{aligned} \int p'(G_i X) p'(X^{-1} G_j) d\mu_H(X) &\sim p''(G_i G_j) \\ \int p''(G_i X) p''(X^{-1} G_j) d\mu_H(X) &\sim p(G_i G_j) \end{aligned} \quad (7)$$

for plaquette subdivisions of all sizes.

Proof: In the directed set $\{\Gamma, \succ\}$ consider two elements Γ and Γ' which differ only in subdivision of a single plaquette from $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$ size (see Fig.2) plus the additional $\frac{a}{2^k}$ plaquettes as explained in the introduction.

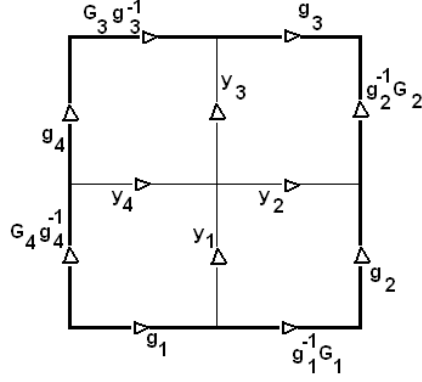


Figure 2: Subdivision of one plaquette

The consistency condition is

$$\begin{aligned}
& \frac{1}{Z'} \int p' (g_1^{-1} G_1 g_2 y_2^{-1} y_1^{-1}) p' (y_2 g_2^{-1} G_2 g_3^{-1} y_3^{-1}) p' (y_4 y_3 g_3 G_3^{-1} g_4^{-1}) p' (g_1 y_1 y_4^{-1} g_4 G_4^{-1}) \\
& \prod_{i=1}^4 d\mu_H (g_i) d\mu_H (y_i) d\mu_H (G_i) \prod_{k=1}^8 d\mu_H (G_k) \\
& = \frac{1}{Z} \int p (G_1 G_2 G_3^{-1} G_4^{-1}) \prod_{i=1}^4 d\mu_H (G_i) \tag{8}
\end{aligned}$$

the last factor in the left hand side denoting integration over the additional $\frac{a}{2^k}$ plaquettes. Using centrality of p' , redefining

$$g_1 y_1 = X_1, \quad g_2 y_2^{-1} = X_2, \quad y_3 g_3 = X_3^{-1}, \quad y_4^{-1} g_4 = X_4^{-1} \tag{9}$$

and using invariance of the normalized Haar measure, one may integrate over y_1, y_2, y_3, y_4 and G_k , obtaining for the left hand side of (8)

$$\frac{1}{Z'} \int p' (X_1^{-1} G_1 X_2) p' (X_2^{-1} G_2 X_3) p' (X_3^{-1} G_3^{-1} X_4) p' (X_4^{-1} G_4^{-1} X_1) \prod_{i=1}^4 d\mu_H (X_i) d\mu_H (G_i)$$

Therefore if there is a sequence of central functions p', p'', p satisfying the proportionality relations

$$\begin{aligned} \int p'(G_i X) p'(X^{-1} G_j) d\mu_H(X) &\sim p''(G_i G_j) \\ \int p''(G_i X) p''(X^{-1} G_j) d\mu_H(X) &\sim p(G_i G_j) \end{aligned} \quad (10)$$

the consistency condition (8) would be satisfied, with the proportionality constant absorbed in the overall measure normalization. Then a measure would exist in the projective limit, because all elements in the directed set $\{\Gamma, \succ\}$ may be reached by one-plaquette subdivisions.

If $p(U_\square)$ is a constant, $d\mu_{\mathcal{A}_\Gamma}$ is factorizable and the consistency condition is trivially satisfied. $d\mu_{\mathcal{A}_\Gamma}$ would be the Ashtekar-Lewandowski measure for generalized connections [4] [5]. A nontrivial solution that satisfies the consistency condition (8) is the choice of $p(U_\square)$ as the heat kernel

$$K(g, \beta) = \sum_{\lambda \in \Lambda^+} d_\lambda e^{-c(\lambda)\beta} \chi_\lambda(g) \quad (11)$$

with

$$\begin{aligned} \beta &\rightarrow \beta' = \frac{\beta}{4} \\ \beta &\rightarrow \beta'' = \frac{\beta}{2} \end{aligned} \quad (12)$$

β'', β' and β being the constants associated to p'', p' and p . In (11), $g \in \mathbb{G}$, $\beta \in \mathbb{R}^+$, Λ^+ is the set of highest weights, d_λ and $\chi_\lambda(\cdot)$ the dimension and character of the λ -representation and $c(\lambda)$ the spectrum of the Laplacian $\Delta_G := \sum_{i=1}^n \chi_i^2$, $\{\chi_i\}$ being a basis for the Lie algebra of \mathbb{G} .

Finally, one writes for the measure on the lattice Γ

$$d\mu_{\mathcal{A}_\Gamma} = \frac{1}{Z_\Gamma} \prod_{edges} d\mu_H(g_l) \prod_{plaquettes} \sum_{\lambda \in \Lambda^+} d_\lambda e^{-c(\lambda)\beta} \chi_\lambda(g_p) \quad (13)$$

and the consistency condition (4) being satisfied, a measure is also defined on the projective limit lattice, that is, on the projective limit generalized connections \mathcal{A} .

This measure has the required naive continuum limit, both for abelian and non-abelian theories (see [1]). Furthermore by defining infinite-dimensional

test functionals and distributions, a projective triplet was constructed which provides a framework to develop an infinite-dimensional calculus over the hypercubical lattice. In particular, this step is necessary to give a meaning to the density $p(\mathcal{A}_\Gamma)$ in the $\beta \rightarrow 0$ limit, where $p(\mathcal{A}_\Gamma)$ would no longer be a continuous function. Thus $p(\mathcal{A}_\Gamma)$, a density that multiplies the Ashtekar-Lewandowski measure [4] [5] [6], gains a distributional meaning in the framework of the projective triplet.

A theory being completely determined whenever its measure is specified, the construction in [1] provides a rigorous specification of a projective limit Yang-Mills theory for gauge fields over a compact group. Some of the consequences of this specification were already discussed in [1]. Here one analyses the nature of the mass gap which follows from the measure specification.

3 The mass gap

The experimental phenomenology of subnuclear physics provides evidence for the short range of strong interactions. Therefore, if unbroken non-abelian Yang-Mills is the theory of strong interactions, the Hamiltonian, associated to its measure, should have a positive mass gap. This important physical question has been addressed in different ways by several authors. An interesting research approach [7] [8] considers the Riemannian geometry of the (lattice) gauge-orbit space to compute the Ricci curvature. The basic inspiration for this approach is the Bochner-Lichnérowicz [9] [10] inequality which states that if the Ricci curvature is bounded from below, then so is the first non-zero eigenvalue of the Laplace-Beltrami operator. The Laplace-Beltrami operator differs from the Yang-Mills Hamiltonian in that it lacks the chromo-magnetic term, but the hope is that in the relevant physical limit the chromo-electric term dominates the bound. An alternative possibility would be to generalize the Bochner-Lichnérowicz inequality.

Other approaches are based on attempts to solve the Dyson-Schwinger equation (see for example [11] [12] [13]) on a set of exact solutions to the classical Yang-Mills theory [14] or on the ellipticity of the energy operator of cut-off Yang-Mills [15] [16].

Once a consistent Euclidean Yang-Mills measure is obtained, the nature of the mass gap may be found either by computing the distance dependence of the correlation of two local operators or from the lower bound of the spectrum in the corresponding Hamiltonian theory. Here I will use the Hamiltonian

approach using the fact that the Hamiltonian may be obtained from the knowledge of the ground state and the ground state may be obtained from the measure.

One of the axis directions in the lattice is chosen as the time direction. Then, recalling that at each step in the projective limit construction one has a finite-dimensional system, the ground state wave functional $\Psi_0(\theta(0))$ at a particular configuration $\theta(0)$ at time zero is obtained by [17] [18]

$$\begin{aligned} |\Psi_0(\theta(0))|^2 &= \int d\theta \Psi_0^*(\theta) \delta(\theta - \theta(0)) \Psi_0(\theta) \\ &= \int d\mu_{\mathcal{A}}(\theta) \delta(\theta - \theta(0)) \end{aligned} \quad (14)$$

where $\mu_{\mathcal{A}}(\theta)$ is the Euclidean measure and θ and $\theta(0)$ stand respectively for the set of group configurations in the edges and for the set of group configurations in the time-zero slice.

In general the explicit computation of the integral in (14) is not easy. However, to study the nature of the mass gap a full calculation of the ground state wave functional is not required. It uses the interpretation of elliptic operators as generators of a diffusion process [19] [20] and, in the limit of small β , the theory of small perturbations of dynamical systems [21] [22].

The ground state in (14) may be used to develop the usual Hamiltonian approach to lattice theory, for which one uses notations similar to those of Chapter 15 in Ref.[23], the main difference being that instead of constructing the Kogut-Susskind Hamiltonian from the Wilson action, one uses the ground state obtained from the measure. The Hamiltonian will be

$$H_g = \frac{g^2(\beta)}{2\beta} \sum_{l,j,\alpha} \left\{ -\frac{\partial}{\partial \theta_j^\alpha(l)} + L_j^\alpha(l) \right\} \left\{ \frac{\partial}{\partial \theta_j^\alpha(l)} + L_j^\alpha(l) \right\} \quad (15)$$

The $\theta_j^\alpha(l)$'s are the Lie algebra coordinates of the group element $\exp(i\theta_j^\alpha(l)\tau_\alpha)$ at each edge l of the time-zero slice of the lattice, the sum is over edges (l), lattice dimensions (j) and Lie algebra generators (α). $g(\beta)$ is a coupling constant to be adjusted consistently to obtain the continuum limit. Recall that from (12) $\beta \rightarrow 0$ as the length of the lattice edges ($\frac{a}{2k}$) goes to zero.

$$L_j^\alpha(l) = -\frac{1}{\Psi_0} \frac{\partial \Psi_0}{\partial \theta_j^\alpha(l)} \quad (16)$$

which, in particular, implies that the ground state energy E_0 is adjusted to zero.

Making the unitary transformation $H_g \rightarrow H'_g = \Psi_0^{-1} H_g \Psi_0$, the ground state becomes the unit function, all states are multiplied by Ψ_0^{-1} and

$$-\beta H'_g = \frac{g^2(\beta)}{2} \sum_{l,j,\alpha} \frac{\partial}{\partial \theta_j^\alpha(l)} \frac{\partial}{\partial \theta_j^\alpha(l)} + \sum_{l,j,\alpha} b_j^\alpha(l) \frac{\partial}{\partial \theta_j^\alpha(l)} \quad (17)$$

with

$$b_j^\alpha(l) = -g^2(\beta) L_j^\alpha(l) = \frac{g^2(\beta)}{2\Psi_0^2} \frac{\partial \ln \Psi_0^2}{\partial \theta_j^\alpha(l)} \quad (18)$$

The second-order elliptic operator in (17) is the generator of the diffusion process

$$d\theta_j^\alpha(l) = b_j^\alpha(l) dt + g(\beta) dW_j^\alpha(l) \quad (19)$$

with drift $b_j^\alpha(l)$ and diffusion coefficient $g(\beta)$. Ψ_0^2 is the invariant measure of this process. The question of existence of a mass gap for the Hamiltonian H'_g is closely related to principal eigenvalue of the Dirichlet problem

$$\begin{aligned} \beta H'_g u &= \lambda u && \text{in } D \\ u &= 0 && \text{in } \partial D \end{aligned} \quad (20)$$

D being a bounded domain and ∂D its boundary. The principal eigenvalue λ_0 , that is, the smallest positive eigenvalue of $\beta H'_g$ has a stochastic representation [24] [22]

$$\lambda_0 = \sup \left\{ \lambda \geq 0; \sup_{\theta \in D} \mathbb{E}_\theta e^{\lambda \tau} < \infty \right\} \quad (21)$$

\mathbb{E}_θ denoting the expectation value for the process started from the θ configuration and τ the time of first exit from the domain D . The validity of this result hinges on the following condition

(C1) The drift b and the diffusion matrix coefficient $\sigma(g(a)\delta_{ij})$ in this case) must be uniformly Lipschitz continuous with exponent $0 < \alpha \leq 1$ and σ positive definite.

(21) is a powerful result which may be used to compute by numerical means the principal eigenvalue for arbitrary values of g ¹. However, a particularly useful situation is the small noise (small g limit). That the small noise

¹See for example Ref. [25]

limit corresponds to the continuum limit of the lattice theory follows from a consistency argument. Under suitable conditions, to be discussed below, the small noise limit of the lowest eigenvalue (the mass gap) of the operator $\beta H'$ is

$$\beta m \sim \exp \left(-\frac{V}{g^2(\beta)} \right) \quad (22)$$

where V is the value of a functional. Hence, for the physical mass gap m to remain fixed when $\beta \rightarrow 0$, it should also be $g(\beta) \rightarrow 0$. Therefore the small noise limit is indeed the continuum limit.

In the small noise limit the mass gap may be obtained from the Wentzell-Freidlin estimates [21] [22]. Given a bounded domain D for the variables $\theta_j^\alpha(l)$ define the functional

$$I_{t_1, t_2}(\chi) = \frac{1}{2} \int_{t_1}^{t_2} \left(\frac{d\chi}{ds} - b(\chi(s)) \right)^2 ds \quad (23)$$

where $\chi(s \in [t_1, t_2])$ is a path from the configuration $\{\theta\}$ to the boundary ∂D of the domain D . Then let

$$I(t, \{\theta\}, \partial D) = \inf_{\chi} I_{0, t}(\chi) \quad (24)$$

be the infimum over all continuous paths that starting from the configuration $\{\theta\}$ hit the boundary ∂D in time less than or equal to t . A path is said to be a *neutral path* if $I(t, \{\theta\}, \partial D) = 0$.

The value of this functional is controlled by the nature of the deterministic dynamical system

$$\frac{d\theta_j^\alpha(l)}{dt} = b_j^\alpha(l) \quad (25)$$

Assume the following additional condition to be fulfilled:

(C2) There are a number r of ω -limit sets K_i of (25) in the domain D , with all points in each set K_i being equivalent for the functional I , that is, $I(t, x, y) = 0$ if both $x, y \in K_i$ and $b \bullet \nu > 0$, ν being the inward normal to ∂D .

Then [19] [22] with

$$V_i = \inf I(t, x, \partial D) \quad \text{for } x \in K_i \quad (26)$$

and

$$\begin{aligned} V_* &= \max(V_1, \dots, V_r) \\ V^* &= \min(V_1, \dots, V_r) \end{aligned}$$

the lowest non-zero eigenvalue λ_0 satisfies

$$\begin{aligned}\lim_{g \rightarrow 0} (-g^2 \ln \lambda_0(g)) &\leq V^* \\ \lim_{g \rightarrow 0} (-g^2 \ln \lambda_0(g)) &\geq V_*\end{aligned}$$

In particular if there is only one V

$$\lambda_0(g) = \beta m(g) \asymp \exp\left(-\frac{V}{g^2(\beta)}\right) \quad (27)$$

the symbol \asymp meaning logarithmic equivalence in the sense of large deviation theory. If the drift is the gradient of a function, as in (18), the quasi-potential V is simply obtained from the difference of the function at the ω -limit set and the minimum at the boundary.

For details on the theory of small perturbations of dynamical systems as applied to the small β limit of lattice theory refer also to [26] where this technique was applied to an approximate ground state functional. Also [27] [28] [29] [30] provide details on how the ground state measure provides a complete specification of quantum theories both for local and non-local potentials.

Now the existence of a mass gap associated to the Hamiltonian (17), obtained from the measure (13) by (14), hinges on checking the above conditions **(C1)** and **(C2)**. Inserting (13) into (14) one obtains

$$|\Psi_0(g_l(0))|^2 = \int \prod_{edges} d\mu_H(g_l) \delta(g_l - g_l(0)) \prod_{plaquettes} \sum_{\lambda \in \Lambda^+} d_\lambda e^{-c(\lambda)\beta} \chi_\lambda(g_p) \quad (28)$$

g_l being the group element associated to the edges and g_p those associated to the ordered product of group elements around a plaquette, $|\Psi_0(g_l(0))|^2$ being a function only of the group elements on the time slice. For practical calculations one makes a global lattice gauge fixing in (28) but for the present considerations this is not important.

In (28) the only free variables are the edge variables in the time slice or, more precicely, the angles of the maximal torus of the group element associated to the corresponding plaquettes. Smoothness of the heat kernel implies that the Leibnitz rule for derivation under the integral can be applied and the drift $b_j^\alpha(l)$ in (25) is also a smooth function. Therefore condition

(C1) is satisfied. As for condition (C2) one knows that the heat kernel satisfies the following two-sided Gaussian estimate

$$\frac{1}{|B(e, \beta^{\frac{1}{2}})|} c_1 \exp\left(\frac{-d^2(g)}{c_2 \beta}\right) \leq K(g, \beta) \leq \frac{1}{|B(e, \beta^{\frac{1}{2}})|} c_3 \exp\left(\frac{-d^2(g)}{c_4 \beta}\right) \quad (29)$$

$d(g)$ being the Carnot-Carathéodory distance of the group element g to the identity e and $|B(e, \beta^{\frac{1}{2}})|$ is the volume of a ball of radius $\beta^{\frac{1}{2}}$ centered at e [31] [32]. The estimate (29) holds if and only if

(A) the volume growth has the doubling property

$$\forall x \in \mathbb{G}, \forall r > 0, |B(x, 2r)| \leq c |B(x, r)|$$

(B) there is a constant γ such that

$$\forall x \in \mathbb{G}, \forall r > 0, \int_{B(x, r)} |f - Av_{B(x, r)} f|^2 dx \leq \gamma r^2 \int_{B(x, 2r)} |\nabla f|^2$$

$Av_{B(x, r)} f$ being the average of f over the ball $B(x, r)$. In particular if \mathbb{G} is unimodular (B) holds.

For a compact group (A) and (B) being satisfied, the two-sided estimate (29) holds. Therefore the dynamical system (25) has only one ω -limit set, the group identity, and one is in the situation of Eq.(27), V being obtained from the difference of the heat kernel at the identity and at the boundary of the domain. In conclusion:

Theorem: If \mathbb{G} is a compact group, the Hamiltonian (19) obtained from the heat-kernel measure has a positive mass gap in the $\beta \rightarrow 0$ limit, in the sense of Eq.(27).

The existence of the projective limit measure and the projective triplet made in (I), as well as the characterization of the nature of the mass gap obtained here, provide a consistent construction of pure Yang-Mills. Of course, to scale up these results to a full understanding of QCD the role of fermions as well as of the non-generic strata [33] would be required. In particular to clarify the importance of these strata for the structure of low-lying excitations.

References

- [1] R. Vilela Mendes; *An infinite-dimensional calculus for generalized connections in hypercubic lattices*, J. Math. Phys. 52 (2011) 052304.
- [2] J. Kisynski; *On the generation of tight measures*, Studia Math. 30 (1968) 141-151.
- [3] K. Maurin; *General eigenfunction expansions and unitary representations of topological groups*, PWN - Polish Scient. Publ., Warszawa 1968.
- [4] A. Ashtekar and J. Lewandowski; *Differential geometry on the space of connections via graphs and projective limits*, J. Geom. Phys. 17 (1995) 191-230.
- [5] A. Ashtekar and J. Lewandowski; *Projective techniques and functional integration for gauge theories*, J.Math. Phys. 36 (1995) 2170-2191.
- [6] C. Fleischhack; *On the support of physical measures in gauge theories*, arXiv:math-ph/0109030.
- [7] M. S. Laufer and P. Orland; *The metric of Yang-Mills orbit space on the lattice*, Phys.Rev. D88 (2013) 065018
- [8] M. S. Laufer; *The Geometry of Lattice-Gauge-Orbit Space*, Ph. D. Thesis The City University of New York, 2011.
- [9] S. Bochner; *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. 52 (1946) 776-797.
- [10] A. Lichnérowicz; *Géométrie des groupes de transformations*, Dunod, Paris 1958.
- [11] R. Alkofer, A. Hauck, L. von Smekal; *Infrared Behavior of Gluon and Ghost Propagators in Landau Gauge QCD*, Physical Review Letters 79 (1997) 3591-3594.
- [12] V. Gogokhia; *How to demonstrate a possible existence of a mass gap in QCD*, arXiv:hep-th/0604095v4
- [13] B. Holdom; *Soft asymptotics with mass gap*, Physics Letters B 728 (2014) 467-471.

- [14] M. Frasca; *Exact solutions for classical Yang-Mills fields*, arXiv:1409.2351
- [15] A. Dynin; *Quantum Yang-Mills-Weyl dynamics in the Schroedinger paradigm*, Russian Journal of Mathematical Physics 21 (2014) 169–188.
- [16] A. Dynin; *On the Yang-Mills Mass Gap Problem*, Russian Journal of Mathematical Physics 21 (2014) 326–328.
- [17] G. C. Rossi and M. Testa; *Ground State Wave Function from Euclidean Path Integral*, Annals of Physics 148 (1983) 144–167.
- [18] E. Fradkin; *Wave functionals for field theories and path integrals*, Nuclear Physics B389 (1993) 587–600.
- [19] A. Friedman; *Stochastic differential equations and applications, vol. 2*, Academic Press, New York 1976.
- [20] M. Freidlin; *Markov processes and differential equations: Asymptotic problems*, Birkhäuser, Basel 1996.
- [21] A. D. Wentzell and M. I. Freidlin; *On small random perturbations of dynamical systems*, Russian Math. Surveys 25 (1970) 1–55.
- [22] M. I. Freidlin and A. D. Wentzell; *Random perturbations of dynamical systems*, Springer, Berlin 2012.
- [23] M. Creutz; *Quarks, gluons and lattices*, Cambridge U. P., Cambridge 1983.
- [24] R. Z. Khas'minskii; *On positive solutions of the equation $\mathfrak{U}u + V \cdot u = 0$* , Theory Probab. Appl. 4 (1959) 309–318.
- [25] S. M. Eleutério and R. Vilela Mendes; *Numerical predictions from a stochastic model for $SU(2)$ lattice gauge fields*, Phys. Lett. B173 (1986) 332–336.
- [26] R. Vilela Mendes; *Stochastic processes and the non-perturbative structure of the QCD vacuum*, Z. Phys. C - Particles and Fields 54 (1992) 273–281.

- [27] S. Albeverio, R. Høegh-Krohn and L. Streit; *Energy Forms, Hamiltonians, and Distorted Brownian Paths*, J. Math. Phys. 18 (1977) 907-917.
- [28] S. Albeverio, R. Høegh-Krohn and L. Streit; *Regularization of Hamiltonians and Processes*, J. Math. Phys. 21 (1980) 1636-1642.
- [29] L. Streit; *Energy forms: Schroedinger theory, processes*, Physics Reports 77 (1981) 363-375.
- [30] R. Vilela Mendes; *Reconstruction of dynamics from an eigenstate*, J. of Math. Phys. 27 (1986) 178-184.
- [31] L. Saloff-Coste; *Aspects of Sobolev-type inequalities*, Cambridge Lect. Notes 289, Cambridge Univ. Press, Cambridge 2002.
- [32] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon; *Analysis and geometry on groups*, Cambridge Tracts on Math. 100, Cambridge Univ. Press, Cambridge 1992.
- [33] R. Vilela Mendes; *Stratification of the orbit space in gauge theories. The role of nongeneric strata*, J. Phys. A: Math. Gen. 37 (2004) 11485-11498.